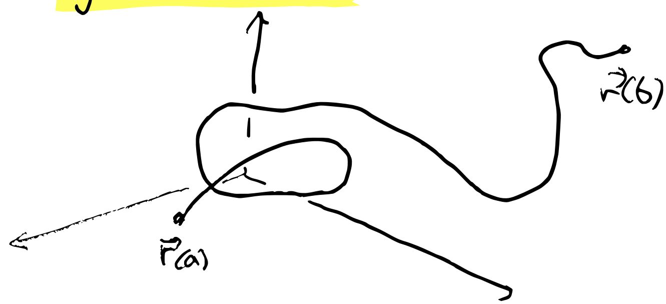
Arc length

Now we can start doing "calculus" for parametric curves. Given a parametric curve $\vec{r}(t)$, $a \le t \le b$,

What is the "length" of the curve?



$$\boxed{\text{Arclength} = \int_a^b |\vec{r}\,'(t)|\,dt}$$

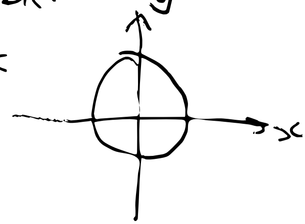
This makes sense, because you are adding the infinitesimal lengths to get the total length, the arclength.

$\vec{r}(b) - \vec{r}(a) =$ "the total vector" $= \int_a^b \vec{r}\,'(t)\,dt$ (vector)

arclength $=$ "the total length" $= \int_a^b |\vec{r}\,'(t)|\,dt$ (scalar)

<u>Example</u> Let's check that the circumference of a circle with radius r is $2\pi r$

$\vec{r}(t) = \langle r\cos t, r\sin t \rangle$, $0 \le t \le 2\pi$



$\Rightarrow \vec{r}\,'(t) = \langle -r\sin t, r\cos t \rangle$

$\Rightarrow |\vec{r}\,'(t)| = \sqrt{(-r\sin t)^2 + (r\cos t)^2}$

$= \sqrt{r^2\sin^2 t + r^2\cos^2 t} = \sqrt{r^2} = r$

$\Rightarrow$ circumference $= \int_0^{2\pi} r\, dt = 2\pi r.$

One interesting aspect of this formula is that the formula gives the arclength regardless of the parametrization.

<u>Example</u> We can chose a weird parametrization

$\vec{r}(t) = \langle r\cos(t^2), r\sin(t^2) \rangle$, $0 \le t \le \sqrt{2\pi}$, which still represents a circle.

$\Rightarrow \vec{r}\,'(t) = \langle -2rt\sin(t^2), 2rt\cos(t^2) \rangle$

$\Rightarrow |\vec{r}\,'(t)| = \sqrt{(-2rt\sin(t^2))^2 + (2rt\cos(t^2))^2} = \sqrt{4r^2t^2(\sin^2(t^2) + \cos^2(t^2))}$

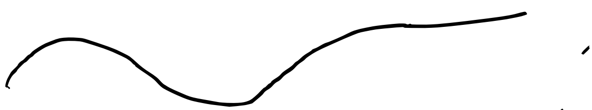
$= \sqrt{4r^2t^2} = 2r|t|$, and

arclength $= \int_0^{\sqrt{2\pi}} 2r|t|\,dt = \int_0^{\sqrt{2\pi}} 2rt\,dt = rt^2\Big|_0^{\sqrt{2\pi}} = 2\pi r$!

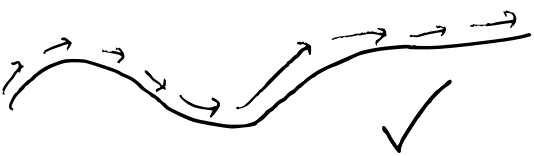
In expressing a curve with some parametrization,

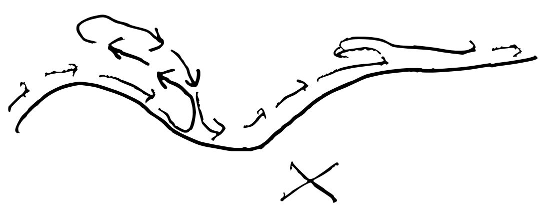
it is important to not change the direction.

Namely, given the path

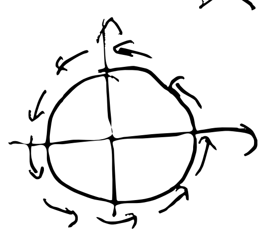


you are allowed to go in either direction with any speed, but you shouldn't change your direction once you start moving.



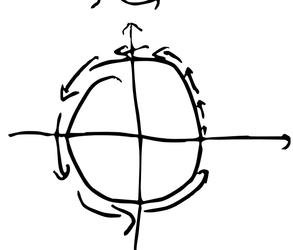


Thus, $\langle r\cos t, r\sin t\rangle$
$0 \le t \le 2\pi$



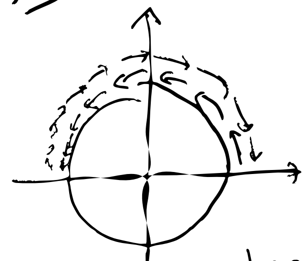
are ok,

$\langle r\cos(t^2), r\sin(t^2)\rangle$
$0 \le t \le \sqrt{2\pi}$



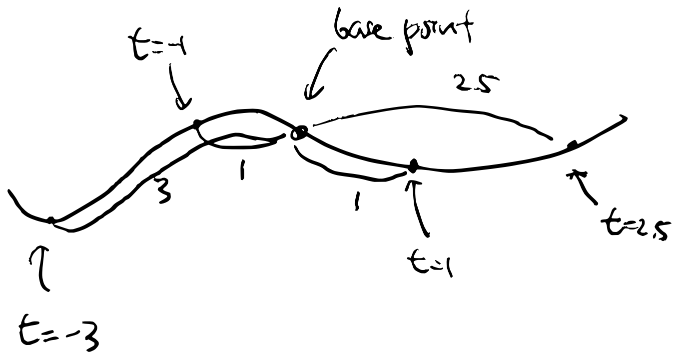
but $\langle r\cos(\pi\sin(t)), r\sin(\pi\sin(t))\rangle$

$0 \le t \le \pi$



is not OK, even though this parametrization has the same start/end points.

It will be useful to have a standard way of parametrizing a curve. We can use the notion of **arclength**:



Namely, the time corresponds to the arclength from the base point! This is called the arclength parametrization.

Definition $\vec{r}(t)$ is an arclength parametrization if $|\vec{r}\,'(t)| = 1$.

3 Steps of finding arclength parametrization.

Step 1. Find the arclength function $l(t)$.

If the start point is $t=a$, $l(t)$ should be

$$\boxed{\begin{array}{ll} l(t)=\int_a^t |\vec{r}\,'(t)|\,dt & \text{(increasing } t \text{ direction)} \\ l(t)=\int_t^a |\vec{r}\,'(t)|\,dt & \text{(decreasing } t \text{ direction)} \end{array}}$$

Step 2 Express t in terms of l.

Step 3 Put the expression from Step 2 to $\vec{r}(t)$ to obtain the arclength parametrization $\vec{r}(l)$.

Example Find the arclength parametrization of $\vec{r}(t)=\langle \cos(t^3), \sin(t^3), t^3\rangle$ starting from $(1,0,0)$ to the direction of increasing t.

Sol Note that $(1,0,0)$ corresponds to $t=0$,

$$\vec{r}\,'(t)=\langle -3t^2\sin(t^3), 3t^2\cos(t^3), 3t^2\rangle$$

$$|\vec{r}\,'(t)|=\sqrt{(-3t^2\sin(t^3))^2+(3t^2\cos(t^3))^2+(3t^2)^2}$$

$$=\sqrt{9t^4(\sin^2(t^3)+\cos^2(t^3)+1)}$$

$$=\sqrt{18t^4}=3\sqrt{2}\,t^2$$

Step 1 The arclength function is

$$\ell(t) = \int_0^t |\vec{r}\,'(t)|\,dt = \int_0^t 3\sqrt{2}\,t^2\,dt = \sqrt{2}\,t^3.$$

Step 2 Since $\ell = \sqrt{2}\,t^3$, $t = \sqrt[3]{\dfrac{\ell}{\sqrt{2}}}$.

Step 3 We plug this back into $\vec{r}(t)$ and get

$$\boxed{\vec{r}(\ell) = \left\langle \cos\left(\frac{\ell}{\sqrt{2}}\right), \sin\left(\frac{\ell}{\sqrt{2}}\right), \frac{\ell}{\sqrt{2}} \right\rangle}$$

Sanity check:

$$\vec{r}\,'(\ell) = \left\langle -\frac{\sin\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}, \frac{\cos\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$|\vec{r}\,'(\ell)| = \sqrt{\left(-\frac{\sin\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}\right)^2 + \left(\frac{\cos\left(\frac{\ell}{\sqrt{2}}\right)}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2}\left(\sin^2\left(\frac{\ell}{\sqrt{2}}\right) + \cos^2\left(\frac{\ell}{\sqrt{2}}\right) + 1\right)} = \sqrt{1} = 1.$$